

## GLOBAL APPROXIMATION OF CONVEX FUNCTIONS

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*Dedicated to the memory of Robb Fry*

ABSTRACT. Let  $U \subseteq \mathbb{R}^n$  be open and convex. We show that every (not necessarily Lipschitz or strongly) convex function  $f : U \rightarrow \mathbb{R}$  can be approximated by real analytic convex functions, uniformly on all of  $U$ . In doing so we provide a technique which transfers results on uniform approximation on bounded sets to results on uniform approximation on unbounded sets, in such a way that not only convexity and  $C^k$  smoothness, but also local Lipschitz constants, minimizers, order, and strict or strong convexity, are preserved. This transfer method is quite general and it can also be used to obtain new results on approximation of convex functions defined on Riemannian manifolds or Banach spaces. We also provide a characterization of the class of convex functions which can be uniformly approximated on  $\mathbb{R}^n$  by strongly convex functions. Finally, we give some counterexamples showing that  $C^0$ -fine approximation of convex functions by smooth convex functions is not possible on  $\mathbb{R}^n$  whenever  $n \geq 2$ .

## 1. INTRODUCTION AND MAIN RESULTS

Two important classes of functions in analysis are those of Lipschitz functions and convex functions  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . Although these functions are almost everywhere differentiable (or even a.e. twice differentiable in the convex case), it is sometimes useful to approximate them by smooth functions which are Lipschitz or convex as well.

In the case of a Lipschitz function  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , this can easily be done as follows: by considering the function  $x \mapsto \inf_{y \in U} \{f(y) + L|x - y|\}$  (where  $L = \text{Lip}(f)$ , the Lipschitz constant of  $f$ ), which is a Lipschitz extension of  $f$  to all of  $\mathbb{R}^n$  having the same Lipschitz constant, one can assume  $U = \mathbb{R}^n$ . Then, by setting  $f_\varepsilon = f * H_\varepsilon$ , where  $H_\varepsilon(x) = \frac{1}{(4\pi\varepsilon)^{n/2}} \exp(-|x|^2/4\varepsilon)$  is the heat kernel, one obtains real analytic Lipschitz functions (with the same Lipschitz constants as  $f$ ) which converge to  $f$  uniformly on all of  $\mathbb{R}^n$  as  $\varepsilon \searrow 0$ . If one replaces  $H_\varepsilon$  with any approximate identity  $\{\delta_\varepsilon\}_{\varepsilon>0}$  of class  $C^k$ , one obtains  $C^k$  Lipschitz approximations. Moreover, if  $\delta_\varepsilon \geq 0$  and  $f$  is convex, then the functions  $f_\varepsilon$  are convex as well.

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However, if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex but not globally Lipschitz, the convolutions  $f * H_\varepsilon$  may not be well defined or, even when they are well defined, they do not converge to  $f$  uniformly on  $\mathbb{R}^n$ . On the other hand, the convolutions  $f * \delta_\varepsilon$  (where  $\delta_\varepsilon = \varepsilon^{-n}\delta(x/\varepsilon)$ ,  $\delta \geq 0$  being a  $C^\infty$  function with bounded support and  $\int_{\mathbb{R}^n} \delta = 1$ ) are always well defined, but they only provide uniform approximation of  $f$  on *bounded* sets. Now, partitions of unity cannot be used to glue these local convex approximations into a global approximation, because they do not preserve convexity. To see why this is so, let us consider the simple case of a  $C^2$  convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , to be approximated by  $C^\infty$  convex functions. Take two bounded intervals  $I_1 \subset I_2$ , and  $C^\infty$  functions  $\theta_1, \theta_2 : \mathbb{R} \rightarrow [0, 1]$  such that  $\theta_1 + \theta_2 = 1$  on  $\mathbb{R}$ ,  $\theta_1 = 1$  on  $I_1$ , and  $\theta_2 = 1$  on  $\mathbb{R} \setminus I_2$ . Given  $\varepsilon_j > 0$  one may find  $C^\infty$  convex functions  $g_j$  such that  $\max\{|f - g_j|, |f' - g_j'|, |f'' - g_j''|\} \leq \varepsilon_j$  on  $I_j$ . If  $g = \theta_1 g_1 + \theta_2 g_2$  one has

$$g'' = g_1''\theta_1 + g_2''\theta_2 + 2(g_1' - g_2')\theta_1' + (g_1 - g_2)\theta_1''.$$

If  $f'' > 0$  on  $I_2$  then by choosing  $\varepsilon_i$  small enough one can control this sum and get  $g'' \geq 0$ , but if the  $g_i'' = 0$  vanish somewhere there is no way to do this (even if we managed to have  $g_2 \geq g_1$  and  $g_2' \geq g_1'$ , as  $\theta_1''$  must change signs).

In [12, 13, 14] Greene and Wu studied the question of approximating a convex function defined on a (finite-dimensional) Riemannian manifold  $M^1$ , and they showed that if  $f : M \rightarrow \mathbb{R}$  is strongly convex (in the sense of the following definition), then for any number  $\varepsilon > 0$  one can find a  $C^\infty$  strongly convex function  $g$  such that  $|f - g| \leq \varepsilon$  on all of  $M$ .

**Definition 1.** A  $C^2$  function  $\varphi : M \rightarrow \mathbb{R}$  is called strongly convex if its second derivative along any nonconstant geodesic is strictly positive everywhere on the geodesic. A (not necessarily smooth) function  $f : M \rightarrow \mathbb{R}$  is said to be strongly convex provided that for every  $p \in M$  and every  $C^\infty$  strongly convex function  $\varphi$  defined on a neighborhood of  $p$  there is some  $\varepsilon > 0$  such that  $f - \varepsilon\varphi$  is convex on the neighborhood.<sup>2</sup>

This solves the problem when the given function  $f$  is strongly convex. However, as Greene and Wu pointed out, their method cannot be used when  $f$  is not strongly convex. This is inconvenient because strong convexity is a

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<sup>1</sup>In Riemannian geometry convex functions have been used, for instance, in the investigation of the structure of noncompact manifolds of positive curvature by Cheeger, Greene, Gromoll, Meyer, Siyohama, Wu and others, see [15, 7, 10, 11, 13, 14]. The existence of global convex functions on a Riemannian manifold has strong geometrical and topological implications. For instance [10], every two-dimensional manifold which admits a global convex function that is locally nonconstant must be diffeomorphic to the plane, the cylinder, or the open Möbius strip.

<sup>2</sup>We warn the reader that, in Greene and Wu's papers, what we have just defined as strong convexity is called strict convexity. We have changed their terminology since we will be mainly concerned with the case  $M = \mathbb{R}^n$ , where one traditionally defines a strictly convex function as a function  $f$  satisfying  $f((1-t)x + ty) < (1-t)f(x) + tf(y)$  if  $0 < t < 1$ .

very strong condition: for instance, the function  $f(x) = x^4$  is strictly convex, but not strongly convex on any neighborhood of 0. However, as shown by Smith in [17], this is a necessary condition in the general Riemannian setting: for each  $k = 0, 1, \dots, \infty$ , there exists a flat Riemannian manifold  $M$  such that on  $M$  there is a  $C^k$  convex function which cannot be globally approximated by a  $C^{k+1}$  convex function (here  $C^{\infty+1}$  means real analytic). There are no results characterizing the manifolds on which global approximation of convex functions by smooth convex functions is possible. Even in the most basic case  $M = \mathbb{R}^n$ , we have been unable to find any reference dealing with the problem of finding smooth global approximations of (not necessarily Lipschitz or strongly) convex functions.

The main purpose of this note is to prove the following.

**Theorem 1.** *Let  $U \subseteq \mathbb{R}^n$  be open and convex. For every convex function  $f : U \rightarrow \mathbb{R}$  and every  $\varepsilon > 0$  there exists a real-analytic convex function  $g : U \rightarrow \mathbb{R}$  such that  $f - \varepsilon \leq g \leq f$ .*

This result is optimal in several ways, as in general it is not possible to obtain  $C^0$ -fine approximation of convex functions by  $C^1$  convex functions on  $\mathbb{R}^n$  when  $n \geq 2$  (and even in the case  $n = 1$  this kind of approximation is not possible from below); see the counterexamples in Section 7 below.

In showing this theorem we will develop a gluing technique for convex functions which will prove to be useful also in the setting of Riemannian manifolds or Banach spaces.

**Definition 2.** Let  $X$  be  $\mathbb{R}^n$ , or a Riemannian manifold (not necessarily finite-dimensional), or a Banach space, and let  $U \subseteq X$  be open and convex. We will say that a continuous convex function  $f : U \rightarrow \mathbb{R}$  can be approximated from below by  $C^k$  convex functions, uniformly on bounded subsets of  $U$ , provided that for every bounded set  $B$  with  $\text{dist}(B, \partial U) > 0$  and every  $\varepsilon > 0$  there exists a  $C^k$  convex function  $g : U \rightarrow \mathbb{R}$  such that

- (1)  $g \leq f$  on  $U$ , and
- (2)  $f - \varepsilon \leq g$  on  $B$ .

(In the case  $U = X$  we will use the convention that  $\text{dist}(B, \partial U) = \infty$  for every bounded set  $B \subset X$ .)

**Theorem 2** (Gluing convex approximations). *Let  $X$  be  $\mathbb{R}^n$ , or a Riemannian manifold (not necessarily finite-dimensional), or a Banach space, and let  $U \subseteq X$  be open and convex. Assume that  $U = \bigcup_{n=1}^{\infty} B_n$ , where the  $B_n$  are open bounded convex sets such that  $\text{dist}(B_n, \partial U) > 0$  and  $\overline{B_n} \subset B_{n+1}$  for each  $n$ . Assume also that  $U$  has the property that every continuous, convex function  $f : U \rightarrow \mathbb{R}$  can be approximated from below by  $C^k$  convex (resp. strongly convex) functions ( $k \in \mathbb{N} \cup \{\infty\}$ ), uniformly on bounded subsets of  $U$ .*

*Then every continuous convex function  $f : U \rightarrow \mathbb{R}$  can be approximated from below by  $C^k$  convex (resp. strongly convex) functions, uniformly on  $U$ .*

From this result (and from its proof and the known results on approximation on bounded sets) we will easily deduce the following corollaries.

**Corollary 1.** *Let  $U \subseteq \mathbb{R}^n$  be open and convex. For every convex function  $f : U \rightarrow \mathbb{R}$  and every  $\varepsilon > 0$  there exists a  $C^\infty$  convex function  $g : U \rightarrow \mathbb{R}$  such that  $f - \varepsilon \leq g \leq f$ . Moreover  $g$  can be taken so as to preserve local Lipschitz constants of  $f$  (meaning  $\text{Lip}(g|_B) \leq \text{Lip}(f|_{(1+\varepsilon)B})$  for every ball  $B \subset U$ ). And if  $f$  is strictly (or strongly) convex, so can  $g$  be chosen.*

**Corollary 2.** *Let  $M$  be a Cartan-Hadamard Riemannian manifold (not necessarily finite dimensional), and  $U \subseteq M$  be open and convex. For every convex function  $f : U \rightarrow \mathbb{R}$  which is bounded on bounded subsets  $B$  of  $U$  with  $\text{dist}(B, \partial U) > 0$ , and for every  $\varepsilon > 0$  there exists a  $C^1$  convex function  $g : U \rightarrow \mathbb{R}$  such that  $f - \varepsilon \leq g \leq f$ . Moreover  $g$  can be chosen so as to preserve the set of minimizers and the local Lipschitz constants of  $f$ . And, if  $f$  is strictly convex, so can  $g$  be taken.*

One should expect that the above corollary is not optimal (in that approximation by  $C^\infty$  convex functions should be possible).

**Corollary 3.** *Let  $X$  be a Banach space whose dual is locally uniformly convex, and  $U \subseteq X$  be open and convex. For every convex function  $f : U \rightarrow \mathbb{R}$  which is bounded on bounded subsets  $B$  of  $U$  with  $\text{dist}(B, \partial U) > 0$ , and for every  $\varepsilon > 0$  there exists a  $C^1$  convex function  $g : U \rightarrow \mathbb{R}$  such that  $f - \varepsilon \leq g \leq f$ . Moreover  $g$  can be taken so as to preserve the set of minimizers and the local Lipschitz constants of  $f$ . And if  $f$  is strictly convex, so can  $g$  be taken.*

A question remains open whether every convex function  $f$  defined on a separable infinite-dimensional Hilbert space  $X$  which is bounded on bounded sets can be globally approximated by  $C^2$  convex functions (notice that Theorem 2 cannot be combined with the results of [8, 9] on smooth and real analytic approximation of bounded convex bodies in order to give a solution to this problem. Although one can use these results, together with the implicit function theorem, to find smooth convex approximations of  $f$  on a bounded set, the approximating functions obtained by this process are not defined on all of  $X$  and are not strongly convex, hence it is not clear how to extend them to a smooth convex function below  $f$  on  $X$ , or even if this should be possible at all).

Finally, as a byproduct of the proof of Theorem 1 we will also obtain the following characterization of the class of convex functions that can be globally approximated by strongly convex functions on  $\mathbb{R}^n$ .

**Proposition 1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. The following conditions are equivalent:*

- (1)  *$f$  cannot be uniformly approximated by strictly convex functions.*
- (2)  *$f$  cannot be uniformly approximated by strongly convex functions.*

- (3) *There exist  $k < n$ , a linear projection  $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , a convex function  $c : \mathbb{R}^k \rightarrow \mathbb{R}$  and a linear function  $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f = c \circ P + \ell$ .*

## 2. THE GLUING TECHNIQUE

In order to prove Theorem 2 we will use the following.

**Lemma 1** (Smooth maxima). *For every  $\varepsilon > 0$  there exists a  $C^\infty$  function  $M_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$  with the following properties:*

- (1)  $M_\varepsilon$  is convex;
- (2)  $\max\{x, y\} \leq M_\varepsilon(x, y) \leq \max\{x, y\} + \frac{\varepsilon}{2}$  for all  $(x, y) \in \mathbb{R}^2$ .
- (3)  $M_\varepsilon(x, y) = \max\{x, y\}$  whenever  $|x - y| \geq \varepsilon$ .
- (4)  $M_\varepsilon(x, y) = M_\varepsilon(y, x)$ .
- (5)  $\text{Lip}(M_\varepsilon) = 1$  with respect to the norm  $\|\cdot\|_\infty$  in  $\mathbb{R}^2$ .
- (6)  $y - \varepsilon \leq x < x' \implies M_\varepsilon(x, y) < M_\varepsilon(x', y)$ .
- (7)  $x - \varepsilon \leq y < y' \implies M_\varepsilon(x, y) < M_\varepsilon(x, y')$ .
- (8)  $x \leq x', y \leq y' \implies M_\varepsilon(x, y) \leq M_\varepsilon(x', y')$ , with a strict inequality in the case when both  $x < x'$  and  $y < y'$ .

We will call  $M_\varepsilon$  a smooth maximum.

*Proof.* It is easy to construct a  $C^\infty$  function  $\theta : \mathbb{R} \rightarrow (0, \infty)$  such that:

- (1)  $\theta(t) = |t|$  if and only if  $|t| \geq \varepsilon$ ;
- (2)  $\theta$  is convex and symmetric;
- (3)  $\text{Lip}(\theta) = 1$ .

Then it is also easy to check that the function  $M_\varepsilon$  defined by

$$M_\varepsilon(x, y) = \frac{x + y + \theta(x - y)}{2}$$

satisfies the required properties. For instance, let us check properties (5), (6), (7) and (8), which are perhaps less obvious than the others. Since  $\theta$  is 1-Lipschitz we have

$$\begin{aligned} M_\varepsilon(x, y) - M_\varepsilon(x', y') &= \frac{x - x' + y - y' + \theta(x - y) - \theta(x' - y')}{2} \leq \\ &= \frac{(x - x') + (y - y') + |x - x' - y + y'|}{2} = \\ &= \max\{x - x', y - y'\} \leq \max\{|x - x'|, |y - y'|\}, \end{aligned}$$

which establishes (5). To verify (6) and (7), note that our function  $\theta$  must satisfy  $|\theta'(t)| < 1 \iff |t| < \varepsilon$ . Then we have

$$\frac{\partial M_\varepsilon}{\partial x}(x, y) = \frac{1}{2} (1 + \theta'(x - y)) \geq \frac{1}{2} (1 - |\theta'(x - y)|) > 0 \text{ whenever } |x - y| < \varepsilon,$$

while

$$\frac{\partial M_\varepsilon}{\partial x}(x, y) = \frac{1}{2} (1 + \theta'(x - y)) = \begin{cases} 1, & \text{if } x \geq y + \varepsilon, \\ 0, & \text{if } y \geq x + \varepsilon. \end{cases}$$

This implies (6) and, together with (4), also (7) and the first part of (8). Finally, if for instance we have  $x' > x = \max\{x, y\}$  then  $M_\varepsilon(x, y) < M_\varepsilon(x', y)$  by (6), and if in addition  $y' > y$  then  $M_\varepsilon(x', y) \leq M_\varepsilon(x', y')$  by the first part of (8), hence  $M_\varepsilon(x, y) < M_\varepsilon(x', y')$ . This shows the second part of (8).  $\square$

The smooth maxima  $M_\varepsilon$  are useful to approximate the maximum of two functions without losing convexity or other key properties of the functions, as we next see.

**Proposition 2.** *Let  $U \subseteq X$  be as in the statement of Theorem 2,  $M_\varepsilon$  as in the preceding Lemma, and let  $f, g : U \rightarrow \mathbb{R}$  be convex functions. For every  $\varepsilon > 0$ , the function  $M_\varepsilon(f, g) : U \rightarrow \mathbb{R}$  has the following properties:*

- (1)  $M_\varepsilon(f, g)$  is convex.
- (2) If  $f$  is  $C^k$  on  $\{x : f(x) \geq g(x) - \varepsilon\}$  and  $g$  is  $C^k$  on  $\{x : g(x) \geq f(x) - \varepsilon\}$  then  $M_\varepsilon(f, g)$  is  $C^k$  on  $U$ . In particular, if  $f, g$  are  $C^k$ , then so is  $M_\varepsilon(f, g)$ .
- (3)  $M_\varepsilon(f, g) = f$  if  $f \geq g + \varepsilon$ .
- (4)  $M_\varepsilon(f, g) = g$  if  $g \geq f + \varepsilon$ .
- (5)  $\max\{f, g\} \leq M_\varepsilon(f, g) \leq \max\{f, g\} + \varepsilon/2$ .
- (6)  $M_\varepsilon(f, g) = M_\varepsilon(g, f)$ .
- (7)  $\text{Lip}(M_\varepsilon(f, g)|_B) \leq \max\{\text{Lip}(f|_B), \text{Lip}(g|_B)\}$  for every ball  $B \subset U$  (in particular  $M_\varepsilon(f, g)$  preserves common local Lipschitz constants of  $f$  and  $g$ ).
- (8) If  $f, g$  are strictly convex on a set  $B \subseteq U$ , then so is  $M_\varepsilon(f, g)$ .
- (9) If  $f, g \in C^2(X)$  are strongly convex on a set  $B \subseteq U$ , then so is  $M_\varepsilon(f, g)$ .
- (10) If  $f_1 \leq f_2$  and  $g_1 \leq g_2$  then  $M_\varepsilon(f_1, g_1) \leq M_\varepsilon(f_2, g_2)$ .

*Proof.* Properties (2), (3), (4), (5), (6), (7) and (10) are obvious from the preceding lemma. To check (1) and (8), we simply use (10) and convexity of  $f, g$  and  $M_\varepsilon$  to see that, for  $x, y \in U$ ,  $t \in [0, 1]$ ,

$$\begin{aligned} M_\varepsilon(f(tx + (1-t)y), g(tx + (1-t)y)) &\leq \\ M_\varepsilon(tf(x) + (1-t)f(y), tg(x) + (1-t)g(y)) &= \\ M_\varepsilon(t(f(x), g(x)) + (1-t)(f(y), g(y))) &\leq \\ tM_\varepsilon(f(x), g(x)) + (1-t)M_\varepsilon(f(y), g(y)), \end{aligned}$$

and, according to (8) in the preceding lemma, the first inequality is strict whenever  $f, g$  are strictly convex and  $0 < t < 1$ . To check (9), it is sufficient to see that the function  $t \mapsto M_\varepsilon(f, g)(\gamma(t))$  has a strictly positive second derivative at each  $t$ , where  $\gamma(t) = x + tv$  with  $v \neq 0$  (or, in the Riemannian case,  $\gamma$  is a nonconstant geodesic). So, by replacing  $f, g$  with  $f(\gamma(t))$  and  $g(\gamma(t))$  we can assume that  $f$  and  $g$  are defined on an interval  $I \subseteq \mathbb{R}$  on

which we have  $f''(t) > 0, g''(t) > 0$ . But in this case we easily compute

$$\begin{aligned} \frac{d^2}{dt^2} M_\varepsilon(f(t), g(t)) &= \\ \frac{(1 + \theta'(f(t) - g(t))) f''(t) + (1 - \theta'(f(t) - g(t))) g''(t)}{2} + \\ + \frac{\theta''(f(t) - g(t)) (f(t) - g(t))^2}{2} &\geq \\ \geq \frac{1}{2} \min\{f''(t), g''(t)\} &> 0, \end{aligned}$$

because  $|\theta'| \leq 1$  and  $\theta'' \geq 0$ .  $\square$

### Proof of Theorem 2.

Given a continuous convex function  $f : U \rightarrow \mathbb{R}$  and  $\varepsilon > 0$ , we start defining  $f_1 = f$  and use the assumption that  $f_1 - \varepsilon/2$  can be approximated from below by  $C^k$  convex functions, to find a  $C^k$  convex function  $h_1 : U \rightarrow \mathbb{R}$  such that

$$f_1 - \varepsilon \leq h_1 \text{ on } B_1, \text{ and } h_1 \leq f_1 - \frac{\varepsilon}{2} \text{ on } U.$$

We put  $g_1 = h_1$ . Now define  $f_2 = f_1 - \varepsilon$  and find a convex function  $h_2 \in C^k(U)$  such that

$$f_2 - \frac{\varepsilon}{2} \leq h_2 \text{ on } B_2, \text{ and } h_2 \leq f_2 - \frac{\varepsilon}{4} \text{ on } U.$$

Set

$$g_2 = M_{\frac{\varepsilon}{10^2}}(g_1, h_2).$$

By the preceding proposition we know that  $g_2$  is a convex  $C^k$  function satisfying

$$\max\{g_1, h_2\} \leq g_2 \leq \max\{g_1, h_2\} + \frac{\varepsilon}{10^2} \text{ on } U,$$

and

$$g_2(x) = \max\{g_1(x), h_2(x)\} \text{ whenever } |h_1(x) - h_2(x)| \geq \frac{\varepsilon}{10^2}.$$

**Claim 1.** *We have*

$$g_2 = g_1 \text{ on } B_1, \text{ and } f - \varepsilon - \frac{\varepsilon}{2} \leq g_2 \leq f - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2} \text{ on } B_2.$$

Indeed, if  $x \in B_1$ ,

$$g_1(x) \geq f_1(x) - \varepsilon = f_2(x) - \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq h_2(x) + \frac{\varepsilon}{4} \geq h_2(x) + \frac{\varepsilon}{10^2},$$

hence  $g_2(x) = g_1(x)$ , and in particular  $f(x) - \frac{\varepsilon}{2} \geq g_2(x) \geq f(x) - \varepsilon$ . While, if  $x \in B_2 \setminus B_1$  then

$$\begin{aligned} f(x) - \varepsilon - \frac{\varepsilon}{2} &\leq \max\{g_1(x), h_2(x)\} \leq g_2(x) \leq \max\{g_1(x), h_2(x)\} + \frac{\varepsilon}{10^2} \leq \\ \max\{f(x) - \frac{\varepsilon}{2}, f(x) - \varepsilon - \frac{\varepsilon}{4}\} + \frac{\varepsilon}{10^2} &= f(x) - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2}. \end{aligned}$$

This proves the claim.

Next, define  $f_3 = f_2 - \varepsilon/2 = f - \varepsilon - \varepsilon/2$ , find a convex  $C^k$  function  $h_3$  on  $U$  so that

$$f_3 - \frac{\varepsilon}{2^2} \leq h_3 \text{ on } B_3, \text{ and } h_3 \leq f_3 - \frac{\varepsilon}{2^3} \text{ on } U,$$

and set

$$g_3 = M_{\frac{\varepsilon}{10^3}}(g_2, h_3).$$

**Claim 2.** *We have*

$$g_3 = g_2 \text{ on } B_2, \text{ and } f - \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2^2} \leq g_3 \leq f - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2} + \frac{\varepsilon}{10^3} \text{ on } B_3.$$

This is easily checked as before.

In this fashion we can inductively define a sequence of  $C^k$  convex functions  $g_n$  on  $U$  such that

$$g_n = g_{n-1} \text{ on } B_{n-1}, \text{ and}$$

$$f - \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2^2} - \dots - \frac{\varepsilon}{2^{n-1}} \leq g_n \leq f - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2} + \frac{\varepsilon}{10^3} + \dots + \frac{\varepsilon}{10^n} \text{ on } B_n$$

(at each step of the inductive process we define  $f_n = f_{n-1} - \varepsilon/2^{n-2} = f - \varepsilon - \dots - \varepsilon/2^{n-2}$ , we find  $h_n$  convex and  $C^k$  such that  $f_n - \varepsilon/2^{n-1} \leq h_n$  on  $B_n$  and  $h_n \leq f_n - \varepsilon/2^n$  on  $U$ , and we put  $g_n = M_{\varepsilon/10^n}(g_{n-1}, h_n)$ ).

Having constructed a sequence  $g_n$  with such properties, we finally define

$$g(x) = \lim_{n \rightarrow \infty} g_n(x).$$

Since we have  $g_{n+k} = g_n$  on  $B_n$  for all  $k \geq 1$ , it is clear that  $g = g_n$  on each  $B_n$ , which implies that  $g$  is  $C^k$  and convex on  $U$  (or even strongly convex when the  $g_n$  are strongly convex). Besides, for every  $x \in U = \bigcup_{n=1}^{\infty} B_n$  we have

$$f(x) - 2\varepsilon = f(x) - \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n-1}} \leq g(x) \leq f(x) - \frac{\varepsilon}{2} + \sum_{n=2}^{\infty} \frac{\varepsilon}{10^n},$$

hence  $f - 2\varepsilon \leq g \leq f$ .  $\square$

**Remark 1.** From the above proof and from Proposition 2 it is clear that this method of transferring convex approximations on bounded sets to global convex approximations preserves strict and strong convexity, local Lipschitzness, minimizers and order, whenever the given approximations on bounded sets have these properties.

### 3. WHY APPROXIMATING LIPSCHITZ CONVEX FUNCTIONS IS ENOUGH

We will deduce our corollaries by combining Theorem 2 with the known results on approximation of convex functions on bounded sets mentioned in the introduction, and with the following.

**Proposition 3.** *Let  $X$  be  $\mathbb{R}^n$ , or a Cartan-Hadamard manifold (not necessarily finite-dimensional), or a Banach space, and let  $U \subseteq X$  be open and convex. Assume that  $U$  has the property that every Lipschitz convex function*



on  $U$  can be approximated by  $C^k$  convex (resp. strongly convex) functions, uniformly on  $U$ .

Then every convex function  $f : U \rightarrow \mathbb{R}$  which is bounded on bounded subsets  $B$  of  $U$  with  $\text{dist}(B, \partial U) > 0$  can be approximated from below by  $C^k$  convex (resp. strongly convex) functions, uniformly on bounded subsets of  $U$ .

*Proof.* It is well known that a convex function  $f : U \rightarrow \mathbb{R}$  which is bounded on bounded subsets  $B$  of  $U$  with  $\text{dist}(B, \partial U) > 0$  is also Lipschitz on each such subset  $B$  of  $X$ . So let  $B \subset U$  be bounded, open and convex with  $\text{dist}(B, \partial U) > 0$ , put  $L = \text{Lip}(f|_B)$ , and define

$$g(x) = \inf\{f(y) + Ld(x, y) : y \in U\},$$

where  $d(x, y) = \|x - y\|$  in the case when  $X$  is  $\mathbb{R}^n$  or a Banach space, and  $d$  is the Riemannian distance in  $X$  when  $X$  is a Cartan-Hadamard manifold.

**Claim 3.** *The function  $g$  has the following properties:*

- (1)  $g$  is convex on  $X$ .
- (2)  $g$  is  $L$ -Lipschitz on  $X$ .
- (3)  $g = f$  on  $B$ .
- (4)  $g \leq f$  on  $U$ .

These are well known facts in the vector space case, but perhaps not so in the Riemannian setting, so let us say a few words about the proof. Property (4) is obvious. To see that the reverse inequality holds on  $B$ , take  $x \in B$  and a subdifferential  $\zeta \in D^-f(x)$  (we refer to [3, 2] for the definitions and some properties of the Fréchet subdifferential and inf convolution on Riemannian manifolds). We have  $\|\zeta\|_x \leq L$  because  $f$  is  $L$ -Lipschitz on  $B$ . Since  $\exp_x : TX_x \rightarrow X$  is a diffeomorphism, for every  $y \in X$  there exists  $v_y \in TX_x$  such that  $\exp_x(v_y) = y$ . And, because  $t \mapsto f(\exp_x(tv_y))$  is convex, we have  $f(\exp_x(tv_y)) - f(x) \geq \langle \zeta, tv_y \rangle_x$  for every  $t$ , and in particular, taking  $t = 1$ , we get  $f(y) - f(x) \geq \langle \zeta, v_y \rangle_x \geq -\|\zeta\|_x \|v_y\|_x \geq -Ld(x, y)$ . Hence  $f(y) + Ld(x, y) \geq f(x)$  for all  $y \in X$ , and taking the inf we get  $g(x) \geq f(x)$ . Therefore  $g = f$  on  $B$ . Showing (2) is easy (as a matter of fact this is true in every metric space). Finally, to see that  $g$  is convex on  $X$ , one does have to use that  $X$  is a Cartan-Hadamard manifold. We note that in a Cartan-Hadamard manifold  $X$  the distance function  $d : X \times X \rightarrow [0, \infty)$  is globally convex (see for instance [18, V.4.3] and [2, Corollary 4.2]), and that if  $X \times U \ni (x, y) \mapsto F(x, y)$  is convex then  $x \mapsto \inf_{y \in U} F(x, y)$  is also convex on  $X$  (see [2, Lemma 3.1]). Since  $(x, y) \mapsto f(y) + Ld(x, y)$  is convex on  $X \times U$ , this shows (1).

Now, for a given  $\varepsilon > 0$ , by assumption there exists a  $C^k$  convex (resp. strongly convex) function  $\varphi : U \rightarrow \mathbb{R}$  so that  $g - \varepsilon \leq \varphi \leq g$  on  $U$ . Since  $g \leq f$  on  $U$ , and  $g = f$  on  $B$ , this implies that  $\varphi \leq f$  on  $U$ , and  $f - \varepsilon \leq \varphi$  on  $B$ .  $\square$

4. UNIFORM APPROXIMATION OF CONVEX FUNCTIONS ON  $\mathbb{R}^n$ 

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous. As we recalled in the introduction, if  $\delta : \mathbb{R}^n \rightarrow [0, \infty)$  is a  $C^\infty$  function such that  $\delta(x) = 0$  whenever  $\|x\| \geq 1$ , and  $\int_{\mathbb{R}^n} \delta = 1$ , then the functions  $f_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y)\delta_\varepsilon(y)dy$  (where  $\delta_\varepsilon(x) = \varepsilon^{-n}\delta(x/\varepsilon)$ ) are  $C^\infty$  and converge to  $f(x)$  uniformly on every compact set, as  $\varepsilon \searrow 0$ . Moreover, as is well known and easily checked:

- (1) If  $f$  is uniformly continuous then  $f_\varepsilon$  converges to  $f$  uniformly on  $\mathbb{R}^n$ .
- (2) If  $f$  is convex (resp. strictly, or strongly convex), so is  $f_\varepsilon$ .
- (3) If  $f$  is Lipschitz, so is  $f_\varepsilon$ , and  $\text{Lip}(f_\varepsilon) = \text{Lip}(f)$ .
- (4) If  $f$  is locally Lipschitz,  $\text{Lip}(f_{\varepsilon|_B}) = \text{Lip}(f|_{(1+\varepsilon)B})$  for every ball  $B$ .
- (5) If  $f \leq g$  then  $f_\varepsilon \leq g_\varepsilon$ .

Therefore this method provides uniform approximation of Lipschitz convex functions by  $C^\infty$  convex functions, uniformly on  $\mathbb{R}^n$ . By Proposition 3 we then have that every (not necessarily Lipschitz) convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  can be approximated from below by  $C^\infty$  convex functions, uniformly on bounded sets. And by Theorem 2 we get that every convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  can be approximated from below by  $C^\infty$  convex functions, uniformly on  $\mathbb{R}^n$ . Moreover, it is clear that strict (or strong) convexity, local Lipschitzness, and order are preserved by the combination of these techniques.

The case when  $X = U$  is an open convex subset of  $\mathbb{R}^n$  can be treated in a similar way. We consider the open, bounded convex sets  $B_m = \{x \in U : \text{dist}(x, \partial U) > 1/m, \|x\| < m\}$ , so we have  $\overline{B_m} \subset B_{m+1}$ ,  $\text{dist}(B_m, \partial U) > 0$  and  $U = \bigcup_{m=1}^\infty B_m$ . By combining Theorem 2 and Proposition 3, it suffices to show that every Lipschitz, convex function  $f : U \rightarrow \mathbb{R}$  can be approximated by  $C^\infty$  convex functions, uniformly on  $U$ . This can be done as follows: set  $L = \text{Lip}(f)$  and consider

$$g(x) = \inf\{f(y) + L\|x - y\| : y \in U\}, \quad x \in \mathbb{R}^n,$$

which is a Lipschitz, convex extension of  $f$  to all of  $\mathbb{R}^n$ , with  $\text{Lip}(f) = \text{Lip}(g)$ . By using the above argument,  $g$  can be approximated by  $C^\infty$  convex functions, uniformly on  $\mathbb{R}^n$ . In particular,  $f = g|_U$  can be approximated by such functions, uniformly on  $U$ . This proves Corollary 1.

## 5. THE RIEMANNIAN AND THE BANACH CASES

Let us see how one can deduce Corollaries 2 and 3. As in the case of  $\mathbb{R}^n$ , the combination of Theorem 2, Proposition 3 and Remark 1 reduces the problem to showing that every *Lipschitz* convex function  $f : X \rightarrow \mathbb{R}$  (where  $X$  stands for a Cartan-Hadamard manifold or a Banach space whose dual is locally uniformly convex) can be approximated by  $C^1$  convex functions, uniformly on  $X$ . It is well known that this can be done via the inf convolution

of  $f$  with squared distances: the functions

$$f_\lambda(x) = \inf \left\{ f(y) + \frac{1}{2\lambda} d(x, y)^2 : y \in X \right\}$$

are  $C^1$ , convex, Lipschitz (with the same constant as  $f$ ), have the same minimizers as  $f$ , are strictly convex whenever  $f$  is, and converge to  $f$  as  $\lambda \searrow 0$ , uniformly on all of  $X$ . See [19] for a survey on the inf convolution operation in Banach spaces, and [2] for the Cartan-Hadamard case.

## 6. REAL ANALYTIC CONVEX APPROXIMATIONS

Let us finally consider the question about global approximation of convex functions  $f$  by real analytic convex functions on  $\mathbb{R}^n$ . As mentioned in the introduction, real analytic approximations of partitions of unity cannot be employed to glue local approximations into a uniform approximation of  $f$  on all of  $\mathbb{R}^n$ , unless those local approximations are strongly convex. Obviously, Theorem 2 cannot be directly used to this purpose either, because it only provides  $C^\infty$  smoothness.

A natural approach to this problem could be trying to approximate (not necessarily strongly) convex functions by real analytic strongly convex functions on bounded sets, and then gluing all the approximating functions by using a real analytic approximation to a partition of unity. This would lead to cumbersome technical problems, as the involved functions would no longer have bounded supports, so one would have to control an infinite sum of products of functions and their first and second derivatives, and make the second derivatives of the convex functions prevail. An essentially equivalent, but more efficient approach would be the following: first, showing that every convex function can be approximated by  $C^2$  strongly convex functions, and then using Whitney's theorem on  $C^2$ -fine approximation of functions by real analytic functions to conclude. This is what we will try to do.

However, not every convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  can be approximated by strongly convex functions uniformly on  $\mathbb{R}^n$ . For instance, it is not possible to approximate a linear function by strongly convex functions. This impossibility is not related to the fact that a linear function is not strongly convex, but rather to the fact that the range of its subdifferential is a singleton: if one considers  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \max\{ax + b, cx + d\}$ , with  $a \neq c$  (which is piecewise linear, and in particular not even strictly convex), then one immediately realizes that  $f$  can be uniformly approximated by  $C^\infty$  strongly convex functions. Since every convex function on  $\mathbb{R}$  can be approximated by piecewise linear convex functions, a combination of this observation with the gluing technique of Section 2, allows us to solve the problem in the way we set out to do, provided that the given function is not affine. And of course, if it is affine, it already is real analytic, so there is nothing to show.

It  $\mathbb{R}^2$  the situation is more complicated: for instance, the function  $f(x, y) = |x - y|$  is not affine and it cannot be approximated by strongly convex functions. However, up to a linear change of coordinates,  $f(u, v) = |u|$ , so if we construct a real analytic convex approximation  $\theta$  of  $t \mapsto |t|$  on  $\mathbb{R}$  and we define  $g(u, v) = \theta(u)$  then we get a real analytic convex approximation of  $f$ . On the other hand, every function of the form  $h(x_1, x_2) = \max\{a_1x_1 + b_1, c_1x_1 + d_1, a_2x_2 + b_2, c_2x_2 + d_2\}$  (with  $a_i \neq c_i$ ) can be uniformly approximated by strongly convex functions (even though its graph is a finite union of convex subsets of hyperplanes).

We next elaborate on these ideas to show that, given a convex function  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , either we can reduce the problem of approximating  $f$  by real analytic convex functions to some  $\mathbb{R}^k$  with  $k < n$ , or else its graph is supported by a maximum of finitely many  $(n + 1)$ -dimensional corners which besides approximates  $f$  on a given bounded set (and which in turn we will manage to approximate by strongly convex functions).

**Definition 3** (Supporting corners). We will say that a function  $C : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $k$ -dimensional *corner function* on  $\mathbb{R}^n$  if it is of the form

$$C(x) = \max\{\ell_1 + b_1, \ell_2 + b_2, \dots, \ell_k + b_k\},$$

where the  $\ell_j : \mathbb{R}^n \rightarrow \mathbb{R}$  are linear functions such that the functions  $L_j : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  defined by  $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$ ,  $1 \leq j \leq k$ , are linearly independent, and the  $b_j \in \mathbb{R}$ . We will also say that a convex function  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is supported by  $C$  at a point  $x \in U$  provided we have  $C \leq f$  on  $U$  and  $C(x) = f(x)$ .

**Lemma 2.** *If  $C$  is an  $(n + 1)$ -dimensional corner function on  $\mathbb{R}^n$  then  $C$  can be approximated by  $C^\infty$  strongly convex functions, uniformly on  $\mathbb{R}^n$ .*

*Proof.* We will need to use the following variation of the smooth maximum of Lemma 1: given  $\varepsilon, r > 0$ , let  $\beta_{\varepsilon, r} = |\cdot| * H_r + \varepsilon/2$ , where  $H_r(x) = \frac{1}{(4\pi r)^{1/2}} \exp(-x^2/4r)$ . We have  $\beta_{\varepsilon, r}''(t) = 2e^{-t^2/4r}/(4r\pi)^{1/2} > 0$ , so  $\beta_{\varepsilon, r}$  is strongly convex and 1-Lipschitz, and as  $r \rightarrow 0$  we have  $\beta_{\varepsilon, r}(t) \rightarrow |t| + \varepsilon/2$  uniformly on  $t \in \mathbb{R}$ , so we may find  $r = r(\varepsilon) > 0$  such that  $|t| \leq \beta_{\varepsilon, r}(t) \leq |t| + \varepsilon$  for all  $t$ . Put  $\tilde{\theta}_\varepsilon(t) = \beta_{\varepsilon, r(\varepsilon)}(t)$ , and define  $\tilde{M}_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\tilde{M}_\varepsilon(x, y) = \frac{x + y + \tilde{\theta}_\varepsilon(x - y)}{2}.$$

It is clear that  $\tilde{M}_\varepsilon$  satisfies all the properties of Lemma 1 except for (3).

Now let us prove our lemma. Up to an affine change of variables in  $\mathbb{R}^{n+1}$ , the problem is equivalent to showing that the function

$$f(x) = \max\{0, x_1, x_2, \dots, x_n\}$$

can be uniformly approximated on  $\mathbb{R}^n$  by  $C^\infty$  strongly convex functions. We will show that this is possible by induction on  $n$ .

For  $n = 1$ , the function  $f(x) = \max\{x, 0\}$  is Lipschitz, so the convolutions  $f_\varepsilon = f * H_\varepsilon$  are  $C^\infty$ , Lipschitz and converge to  $f$ , uniformly on  $\mathbb{R}$ , as  $\varepsilon \searrow 0$ . Besides, as one can easily compute,

$$f_\varepsilon''(x) = \frac{1}{(4\pi\varepsilon)^{1/2}} e^{-\frac{x^2}{4\varepsilon}} > 0,$$

so the  $f_\varepsilon$  are strongly convex.

Now, suppose that the function  $f(x_1, \dots, x_k) = \max\{0, x_1, \dots, x_k\}$  can be uniformly approximated by  $C^\infty$  smooth strongly convex functions on  $\mathbb{R}^k$ . Then, for a given  $\varepsilon > 0$  we can find  $C^\infty$  strongly convex functions  $g : \mathbb{R}^k \rightarrow \mathbb{R}$  and  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} f(x) &\leq g(x) \leq f(x) + \varepsilon \quad \text{for all } x \in \mathbb{R}^k, \quad \text{and} \\ \max\{t, 0\} &\leq \alpha(t) \leq \max\{t, 0\} + \varepsilon \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Given the function

$$F(x_1, \dots, x_k, x_{k+1}) = \max\{0, x_1, \dots, x_{k+1}\} = \max\{x_{k+1}, f(x_1, \dots, x_k)\},$$

let us define  $G : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  by

$$G(x_1, \dots, x_{k+1}) = \widetilde{M}_\varepsilon(g(x_1, \dots, x_k), \alpha(x_{k+1})).$$

We have  $G \in C^\infty(\mathbb{R}^{k+1})$ , and  $F(x) \leq G(x) \leq F(x) + 2\varepsilon$  for all  $x \in \mathbb{R}^{k+1}$ , so in order to conclude the proof we only have to see that  $G$  is strongly convex. Given  $x, v \in \mathbb{R}^{k+1}$  with  $v \neq 0$ , it is enough to check that the function

$$h(t) := G(x + tv) = \widetilde{M}_\varepsilon(\beta(t), \gamma(t)),$$

where  $\beta(t) = g(x_1 + tv_1, \dots, x_k + tv_k)$  and  $\gamma(t) = \alpha(x_{k+1} + tv_{k+1})$ , satisfies  $h''(t) > 0$ . If  $v_{k+1} \neq 0$  and  $(v_1, \dots, v_k) \neq 0$  then, since  $g$  is strongly convex on  $\mathbb{R}^k$  and  $\alpha$  is strongly convex on  $\mathbb{R}$ , we have  $\beta''(t) > 0$  and  $\gamma''(t) > 0$ , so exactly as in the proof of (9) of Proposition 2 we also get  $h''(t) > 0$ . On the other hand, if for instance we have  $v_{k+1} = 0$  then  $\beta''(t) > 0$  and  $\gamma'(t) = \gamma''(t) = 0$ , so

$$\begin{aligned} \frac{d^2}{dt^2} \widetilde{M}_\varepsilon(\beta(t), \gamma(t)) &= \\ \frac{(1 + \theta'_\varepsilon(\beta(t) - \gamma(t))) \beta''(t) + \theta''_\varepsilon(\beta(t) - \gamma(t)) (\beta(t) - \gamma(t))^2}{2} &> 0, \end{aligned}$$

because  $|\widetilde{\theta}'_\varepsilon| \leq 1$ ,  $\widetilde{\theta}''_\varepsilon > 0$ , and  $\widetilde{\theta}'_\varepsilon(0) = 0$ . Similarly one checks that  $\frac{d^2}{dt^2} \widetilde{M}_\varepsilon(\beta(t), \gamma(t)) > 0$  in the case when  $(v_1, \dots, v_k) = 0 \neq v_{k+1}$ .  $\square$

**Lemma 3.** *Let  $U \subseteq \mathbb{R}^n$  be open and convex,  $f : U \rightarrow \mathbb{R}$  be a  $C^p$  convex function, and  $x_0 \in U$ . Assume that  $f$  is not supported at  $x_0$  by any  $(n+1)$ -dimensional corner function. Then there exist  $k < n$ , a linear projection  $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , a  $C^p$  convex function  $c : P(U) \subseteq \mathbb{R}^k \rightarrow \mathbb{R}$ , and a linear function  $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f = c \circ P + \ell$ .*

*Proof.* If  $f$  is affine the result is obvious. If  $f$  is not affine then there exists  $y_0 \in U$  with  $f'(x_0) \neq f'(y_0)$ . It is clear that  $L_1(x, x_{n+1}) = x_{n+1} - f'(x_0)(x)$  and  $L_2(x, x_{n+1}) = x_{n+1} - f'(y_0)(x)$  are two linearly independent linear functions on  $\mathbb{R}^{n+1}$ , hence  $f$  is supported at  $x_0$  by the two-dimensional corner  $x \mapsto \max\{f(x_0) + f'(x_0)(x - x_0), f(y_0) + f'(y_0)(x - y_0)\}$ . Let us define  $m$  as the greatest integer number so that  $f$  is supported at  $x_0$  by an  $m$ -dimensional corner. By assumption we have  $2 \leq m < n + 1$ . Define  $k = m - 1$ . There exist  $\ell_1, \dots, \ell_{k+1} \in (\mathbb{R}^n)^*$  with  $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$ ,  $j = 1, \dots, k + 1$ , linearly independent in  $(\mathbb{R}^{n+1})^*$ , and  $b_1, \dots, b_{k+1} \in \mathbb{R}$ , so that  $C = \max_{1 \leq j \leq k+1} \{\ell_j + b_j\}$  supports  $f$  at  $x_0$ .

Observe that the  $\{L_j - L_1\}_{j=2}^{k+1}$  are linearly independent in  $(\mathbb{R}^{n+1})^*$ , hence so are the  $\{\ell_j - \ell_1\}_{j=2}^{k+1}$  in  $(\mathbb{R}^n)^*$ , and therefore  $\bigcap_{j=2}^{k+1} \text{Ker}(\ell_j - \ell_1)$  has dimension  $n - k$ . Then we can find linearly independent vectors  $w_1, \dots, w_{n-k}$  such that  $\bigcap_{j=2}^{k+1} \text{Ker}(\ell_j - \ell_1) = \text{span}\{w_1, \dots, w_{n-k}\}$ .

Now, given any  $y \in U$ , if  $\frac{d}{dt}(f - \ell_1)(y + tw_q)|_{t=t_0} \neq 0$  for some  $t_0$  then  $f'(y + t_0 w_q) - \ell_1$  is linearly independent with  $\{\ell_j - \ell_1\}_{j=2}^{k+1}$ , which implies that  $(x, x_{n+1}) \mapsto x_{n+1} - f'(y + t_0 w_q)$  is linearly independent with  $L_1, \dots, L_{k+1}$ , and therefore the function

$$x \mapsto \max\{\ell_1(x) + b_1, \dots, \ell_{k+1}(x) + b_{k+1}, f'(y + t_0 w_q)(x - y - t_0 w_q) + f(y + t_0 w_q)\}$$

is a  $(k + 2)$ -dimensional corner supporting  $f$  at  $x_0$ , which contradicts the choice of  $m$ . Therefore we must have

$$\frac{d}{dt}(f - \ell_1)(y + tw_q) = 0 \quad \text{for all } y \in U, t \in \mathbb{R} \text{ with } y + tw_q \in U, q = 1, \dots, n - k.$$

This implies that

$$(f - \ell_1)(y + \sum_{j=1}^{n-k} t_j w_j) = (f - \ell_1)(y)$$

if  $y \in U$  and  $y + \sum_{j=1}^{n-k} t_j w_j \in U$ . Let  $Q$  be the orthogonal projection of  $\mathbb{R}^n$  onto the subspace  $E := \text{span}\{w_1, \dots, w_{n-k}\}^\perp$ . For each  $z \in Q(U)$  we may define

$$\tilde{c}(z) = (f - \ell_1)(z + \sum_{j=1}^{n-k} t_j w_j)$$

if  $z + \sum_{j=1}^{n-k} t_j w_j \in U$  for some  $t_1, \dots, t_{n-k}$ . It is clear that  $\tilde{c} : Q(U) \rightarrow \mathbb{R}$  is well defined, convex and  $C^p$ , and satisfies

$$f - \ell_1 = \tilde{c} \circ Q.$$

Then, by taking a linear isomorphism  $T : E \rightarrow \mathbb{R}^k$  and setting  $P = TQ$ , we have that  $f = c \circ P + \ell_1$ , where  $c = \tilde{c} \circ T^{-1}$  is defined on  $P(U)$ .  $\square$

Now we can prove Theorem 1. We already know that a convex function  $f : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$  can be uniformly approximated from below by  $C^1$  functions,

so we may assume that  $f \in C^1(U)$ . We will proceed by induction on  $n$ , the dimension of  $\mathbb{R}^n$ .

For  $n = 1$  the result can be proved as follows. Either  $f : U \rightarrow \mathbb{R}$  is affine (in which case we are done) or  $f$  can be supported by a 2-dimensional corner at every point  $x \in U$ . In the latter case, let us consider a compact interval  $I \subset U$ . Given  $\varepsilon > 0$ , since  $f$  is convex and Lipschitz on  $I$  we can find finitely many affine functions  $h_1, \dots, h_m : \mathbb{R} \rightarrow \mathbb{R}$  such that each  $h_j$  supports  $f - \varepsilon$  at some point  $x_j \in I$  and  $f - 2\varepsilon \leq \max\{h_1, \dots, h_m\}$  on  $I$ . By convexity we also have  $\max\{h_1, \dots, h_m\} \leq f - \varepsilon$  on all of  $I$ . For each  $x_j$  we may find a 2-dimensional corner  $C_j$  which supports  $f - \varepsilon$  at  $x_j$ . Since  $f$  is differentiable and convex we have  $h_j = C_j$  on a neighborhood of  $x_j$  and, by convexity, also  $h_j \leq C_j \leq f - \varepsilon$  and  $\max\{C_1, \dots, C_m\} \leq f - \varepsilon$  on  $I$ . And we also have  $f - 2\varepsilon \leq \max\{h_1, \dots, h_m\} \leq \max\{C_1, \dots, C_m\} \leq f - \varepsilon$  on  $I$ . Now apply Lemma 2 to find  $C^\infty$  strongly convex functions  $g_1, \dots, g_m : \mathbb{R} \rightarrow \mathbb{R}$  such that  $C_j \leq g_j \leq C_j + \varepsilon'$ , where  $\varepsilon' := \varepsilon/2m$ , and define  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g = M_{\varepsilon'}(g_1, M_{\varepsilon'}(g_2, M_{\varepsilon'}(g_3, \dots, M_{\varepsilon'}(g_{m-1}, g_m)) \dots))$$

(for instance, if  $m = 3$  then  $g = M_{\varepsilon'}(g_1, M_{\varepsilon'}(g_2, g_3))$ ). By Proposition 2 we have that  $g \in C^\infty(\mathbb{R})$  is strongly convex,

$$\max\{C_1, \dots, C_m\} \leq g \leq \max\{C_1, \dots, C_m\} + m\varepsilon' \leq f - \frac{\varepsilon}{2} \quad \text{on } U,$$

and

$$f - 2\varepsilon \leq \max\{C_1, \dots, C_m\} \leq g \quad \text{on } I.$$

Therefore  $f : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$  can be approximated from below by  $C^\infty$  strongly convex functions, uniformly on compact subintervals of  $U$ . By Theorem 2 and Remark 1 we conclude that, given  $\varepsilon > 0$  we may find a  $C^\infty$  strongly convex function  $h$  such that  $f - 2\varepsilon \leq h \leq f - \varepsilon$  on  $U$ .

Finally, set  $\eta(x) = \frac{1}{2} \min\{h''(x), \varepsilon\}$  for every  $x \in U$ . The function  $\eta : U \rightarrow (0, \infty)$  is continuous, so we can apply Whitney's theorem on  $C^2$ -fine approximation of  $C^2$  functions by real analytic functions to find a real analytic function  $g : U \rightarrow \mathbb{R}$  such that

$$\max\{|h - g|, |h' - g'|, |h'' - g''|\} \leq \eta.$$

This implies that  $f - 3\varepsilon \leq g \leq f$  and  $g'' \geq \frac{1}{2}h'' > 0$ , so  $g$  is strongly convex as well.

Now assume the result is true in  $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^d$ , and let us see that then it is also true in  $\mathbb{R}^{d+1}$ . If there is some  $x_0 \in U$  such that  $f : U \subseteq \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  is not supported at  $x_0$  by any  $(d+2)$ -dimensional corner function then, according to Lemma 3, we can find  $k \leq d$ , a linear projection  $P : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^k$ , a linear function  $\ell : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ , and a  $C^\infty$  convex function  $c : P(U) \rightarrow \mathbb{R}$  such that  $f = c \circ P + \ell$ . By assumption there exists a real analytic convex function  $h : P(U) \subseteq \mathbb{R}^k \rightarrow \mathbb{R}$  so that  $c - \varepsilon \leq h \leq c$ . Then the function  $g = h \circ P + \ell$  is real analytic, convex (though never strongly convex), and satisfies  $f - \varepsilon \leq g \leq f$ .

If there is no such  $x_0$  then one can repeat exactly the same argument as in the case  $n = 1$ , just replacing 2-dimensional corners with  $(d+2)$ -dimensional corners, the interval  $I$  with a compact convex body  $K \subset U$ , and  $\eta$  with

$$\eta(x) = \frac{1}{2} \min\{\varepsilon, \min\{D^2 h(x)(v)^2 : v \in \mathbb{R}^{d+1}, \|v\| = 1\}\},$$

in order to conclude that there exists a real analytic strongly convex  $g : U \rightarrow \mathbb{R}$  such that  $f - \varepsilon \leq g \leq f$  on  $U$ .  $\square$

Incidentally, the above argument also shows Proposition 1 in the case when  $f$  is  $C^1$ . In the general case of a nonsmooth convex function one just needs to take two more facts into account. First, Lemma 3 holds for nonsmooth convex functions (to see this, use the fact that if the range of the subdifferential of a convex function is contained in  $\{0\}$  then the function is constant, see for instance [6, Chapter 1, Corollary 2.7], and apply this to the function  $(t_1, \dots, t_{n-k}) \mapsto (f - \ell_1)(y + \sum_{j=1}^{n-k} t_j w_j)$ ). Second, in the above proof one can use Rademacher's theorem and uniform continuity of  $f$  to see that the  $x_j$  can be assumed to be points of differentiability of  $f$ .

## 7. THREE COUNTEREXAMPLES

In this section we briefly discuss the possibility of approximating a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by smooth convex functions in the  $C^0$ -fine topology. We will see that there is quite a big difference between the cases  $n = 1$  and  $n \geq 2$ .

In the case  $n = 1$  it can be shown that every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  can be approximated by convex real analytic functions in this topology.<sup>3</sup> That is, for every continuous function  $\varepsilon : \mathbb{R} \rightarrow (0, \infty)$  there exists a real analytic convex function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $|f(x) - g(x)| \leq \varepsilon(x)$  for all  $x \in \mathbb{R}$ . However, this approximation cannot be performed from below:

**Example 1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = |x|$ . For every  $C^1$  convex function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g(0) \leq 0$  we have

$$\liminf_{|x| \rightarrow \infty} |f(x) - g(x)| > 0.$$

In particular, if  $\varepsilon : \mathbb{R} \rightarrow (0, \infty)$  is continuous and satisfies  $\lim_{|x| \rightarrow \infty} \varepsilon(x) = 0$  then there is no  $C^1$  convex function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $|x| - \varepsilon(x) \leq g(x) \leq |x|$ .

In two or more dimensions the situation gets much worse:  $C^0$ -fine approximation of convex functions by  $C^1$  convex functions is no longer possible in general.

**Example 2.** For  $n \geq 2$ , let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by  $f(x_1, \dots, x_n) = |x_1|$ , and let  $\varepsilon : \mathbb{R}^n \rightarrow (0, \infty)$  be continuous with  $\lim_{|x| \rightarrow \infty} \varepsilon(x) = 0$ . Then there is no  $C^1$  convex function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f - g| \leq \varepsilon$ .

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<sup>3</sup>We will provide a proof of this statement in a forthcoming paper.



*Proof.* Suppose that such a function  $g$  exists. Since  $|g(x_1, 0, \dots, 0) - |x_1|| \leq \varepsilon(x_1, 0, \dots, 0)$  and  $\lim_{|x_1| \rightarrow \infty} \varepsilon(x_1, 0, \dots, 0) = 0$ , we know from the preceding example that  $g(0) > 0$ . Consider the function  $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  defined by  $h(y) = g(0, y_1, \dots, y_{n-1})$ . We have  $\lim_{|y| \rightarrow \infty} h(y) = 0$  and  $h(0) > 0$ , but this contradicts the fact that  $h$  is convex.  $\square$

Our last example shows that for  $n \geq 2$  there also exist  $C^\infty$  convex functions on  $\mathbb{R}^n$  which cannot be approximated by real analytic convex functions in the  $C^0$ -fine topology.

**Example 3.** Let  $n \geq 2$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by  $f(x_1, \dots, x_n) = \alpha(x_1)$ , where  $\alpha : \mathbb{R} \rightarrow [0, \infty)$  is a  $C^\infty$  convex function such that  $\alpha(t) = 0$  for  $|t| \leq 1$ , and  $\alpha(t) = |t| - 2$  for  $|t| \geq 3$ . Let  $\varepsilon : \mathbb{R}^n \rightarrow (0, \infty)$  be continuous and such that  $\lim_{|x| \rightarrow \infty} \varepsilon(x) = 0$ . Then there is no real analytic convex function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|f - g| \leq \varepsilon$ .

*Proof.* Suppose there exists such a function  $g$ . As in the preceding example, if  $g(t, 0) > \alpha(t)$  for some  $t$  we easily get a contradiction. Therefore  $t \mapsto g(t, 0)$  is everywhere below  $\alpha$ . By convexity, if  $\alpha(t_0) - g(t_0, 0) := s_0 > 0$  for some  $t_0 \geq 3$ , then  $\alpha(t) - g(t, 0) \geq s_0 > 0$  for all  $t \geq t_0$ , and therefore  $g$  cannot approximate  $f$  as required. Hence  $g(t, 0) = t - 2$  for all  $t \geq 3$ , and since  $g$  is real analytic, also  $g(t, 0) = t - 2$  for all  $t \in \mathbb{R}$ . But then  $\lim_{t \rightarrow -\infty} |g(t, 0) - f(t, 0)| = \infty$ , and again  $g$  cannot approximate  $f$ .  $\square$

## 8. APPENDIX: CONVEX FUNCTIONS VS CONVEX BODIES

In this appendix we recall a (somewhat unbalanced) basic relationship between convex functions and convex bodies, regarding approximation. Given a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , if we consider the epigraph  $C$  of  $f$ , which is an unbounded convex body in  $\mathbb{R}^{n+1}$ , we can approximate  $C$  by smooth convex bodies  $D_k$  such that  $\lim_{k \rightarrow \infty} D_k = C$  in the Hausdorff distance. Then it is easy to see (via the implicit function theorem) that the boundaries  $\partial D_k$  are graphs of smooth convex functions  $g_k : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} g_k = f$  uniformly on compact subsets of  $\mathbb{R}^n$ . But when  $f$  is not Lipschitz this convergence is not uniform on  $\mathbb{R}^n$ , as the following example shows.

**Example 4.** Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$ . The epigraph  $C := \{(x, y) : y \geq x^2\}$  is an unbounded convex body, and the set  $D := \{(x, y) : \text{dist}((x, y), C) \leq \varepsilon/2\}$  is a  $C^1$  convex body such that  $C \subset D \subset C + \varepsilon B$ , where  $B$  is the unit ball of  $\mathbb{R}^2$ . Hence  $D$  approximates  $C$  in the Hausdorff distance, and the boundary  $\partial D$  is indeed the graph of a  $C^1$  convex function  $g : \mathbb{R} \rightarrow \mathbb{R}$ . But the function  $g$  does not approximate  $f$  on  $\mathbb{R}$ , because  $\lim_{|x| \rightarrow \infty} |f(x) - g(x)| = \infty$ .

Therefore one cannot employ results on approximation of (unbounded) convex bodies to deduce results on global approximation of convex functions. By contrast, one can use the well known results on global approximation of

Lipschitz convex functions by real analytic convex functions to deduce the following.

**Corollary 4** (Minkowski). *Let  $C \subset \mathbb{R}^n$  be a (not necessarily bounded) convex body. For every  $\varepsilon > 0$  there exists a real analytic convex body  $D$  such that*

$$C \subset D \subset C + \varepsilon B,$$

where  $B$  is the unit ball of  $\mathbb{R}^n$ .

*Proof.* Consider the 1-Lipschitz, convex function  $f : \mathbb{R}^n \rightarrow [0, \infty)$  defined by  $f(x) = \text{dist}(x, C)$ . Using integral convolution with the heat kernel one can produce a real analytic convex (and 1-Lipschitz) function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f - 2\varepsilon/3 \leq g \leq f - \varepsilon/3$  on  $\mathbb{R}^n$ . Define  $D = g^{-1}(-\infty, 0]$ . Since  $g$  is convex and does not have any minimum on  $\partial D = g^{-1}(0)$ , we have  $\nabla g(x) \neq 0$  for all  $x \in \partial D$ , hence  $\partial D$  is a 1-codimensional real analytic submanifold of  $\mathbb{R}^n$ . Because  $f \geq g$ , we have  $C \subset D$ . And if  $x \notin C + \varepsilon B$  then  $f(x) \geq \varepsilon$ , hence  $g(x) - \varepsilon/3 \geq f(x) - \varepsilon \geq 0$ , which implies  $g(x) > 0$ , that is  $x \notin D$ .  $\square$

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